## APPENDIX A: BROWNIAN TRANSLATION AND ROTATION

### 1. Brownian Translation

The derivation of the expression for the mean square distance travelled, or mean square angle rotated, by an object undergoing Brownian motion, is surprisingly simple.

For motion in the $x$-direction, an object of mass $m$ obeys Newton's second law

$$\begin{aligned} \frac{dx}{dt} &= v \\ m\frac{dv}{dt} &= -\beta v + f(t) \end{aligned} \quad \text{(A1)}$$

In Eq. (A1), the force on the object due to the impact of molecules is divided into two pieces. One is a steady viscous force $-\beta v$ (calculation of the constant $\beta$ is discussed in Appendix B), the other is a random force $f(t)$, which is assumed uncorrelated with $x$. We imagine a collection of identical objects undergoing this motion. Each object suffers a different force, but the average force over the collection is $\overline{f}(t) = 0$. We wish to know the mean square distance $\overline{x^2}$ travelled in time $t$, for this collection. We assume that the molecules are in thermal equilibrium, with each other and with the objects, so by the equipartition theorem,

$$\frac{1}{2}m\overline{v^2} = \frac{1}{2}kT, \quad \text{(A2)}$$

where $k$ is Boltzmann's constant and $T$ is the temperature.

Consider the equations for $x^2$ and $xv$, which follow from Eqs. (A1):

$$\begin{aligned} \frac{1}{2}\frac{dx^2}{dt} &= xv \\ m\frac{dxv}{dt} &= mv^2 - \beta xv + xf(t) \end{aligned} \quad \text{(A3)}$$

Upon taking the average, over the collection, of Eqs. (A3), one obtains

$$\begin{aligned} \frac{1}{2}\frac{d\overline{x^2}}{dt} &= \overline{xv} \\ m\frac{d\overline{xv}}{dt} &= m\overline{v^2} - \beta\overline{xv} \end{aligned} \quad \text{(A4)}$$

since $\overline{xf(t)} = \overline{x}\overline{f}(t) = 0$. Surprisingly, this force, which causes the Brownian motion, appears to plays no role in the subsequent mathematics. However, it does play a role: it is responsible for Eq. (A2), as can be seen by calculating $\overline{vf}(t) \neq 0$ (which we shall not do here, as it is not needed).

One readily sees from the second of Eqs. (A4) that $\overline{xv}$ exponentially decays to a constant, so that the right side vanishes,

$$\overline{xv} = \frac{m}{\beta}\overline{v^2} = \frac{kT}{\beta}, \quad \text{(A5)}$$

where Eq. (A2) is utilized in the second step. Putting Eq. (A5) into the first of Eqs. (A4) and integrating, we obtain the desired result:

$$\overline{x^2} = \frac{2kTt}{\beta}. \quad \text{(A6)}$$

It is useful to have an expression for the mean distance $\overline{|x|}$. It can be argued that the particle position probability density distribution is well approximated by a gaussian distribution,

$$P(x) = \frac{1}{\sqrt{2\pi\overline{x^2}}}e^{-x^2/2\overline{x^2}}$$

which yields the result

$$\overline{|x|} = 2\int_0^\infty dx xP(x) = \sqrt{\frac{2}{\pi}\overline{x^2}} \approx .80\sqrt{\overline{x^2}}.$$

It is also useful to have an expression for the mean distance $\overline{|r|}$ travelled when there is motion in two dimensions. Then

$$\overline{|r|} = \int_0^\infty r^2 dr \int_0^{2\pi} d\theta \frac{1}{2\pi\overline{x^2}}e^{-r^2/2\overline{x^2}} = \sqrt{\frac{\pi}{2}\overline{x^2}} \approx 1.25\sqrt{\overline{x^2}}.$$

### 2. Brownian Rotation

For Brownian rotation through angle $\theta$ about an axis, for an object of moment of inertia $I$, the Newtonian equations are

$$\begin{aligned} \frac{d\theta}{dt} &= \omega \\ I\frac{d\omega}{dt} &= -\beta'\omega + \tau(\omega), \end{aligned} \quad \text{(A7)}$$

where the equipartition theorem implies $(1/2)I\omega^2 = (1/2)kT$, and the random torque satisfies $\overline{\tau}(\omega) = 0$. Eqs. (A1) and (A7) are precisely analogous, so the result (A6) in this case becomes

$$\overline{\theta^2} = \frac{2kTt}{\beta'}. \quad \text{(A8)}$$

## APPENDIX B: VISCOUS FORCE AND TORQUE ON A SPHERE AND ELLIPSOID

### 1. Fluid Flow Equations

The derivation of the expression for the viscous force, felt by an object moving with constant velocity through a fluid, is surprisingly complicated. First one finds the fluid velocity and pressure in an equivalent situation, where the object is at rest and the fluid asymptotically flows with constant velocity. Then one finds the force.

Any fluid obeys the conservation of mass equation,

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\rho\mathbf{v}=0.$$

If the fluid is incompressible, then the mass density $\rho$ is constant, so

$$\nabla\cdot\mathbf{v}=0. \tag{B1}$$

The next equation, the Navier-Stokes equation, is Newton's second law for a bit of fluid,

$$\begin{aligned}\rho\frac{D\mathbf{v}}{Dt} &= -\nabla P+\eta\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\Big[\frac{\partial\mathbf{v}}{\partial x_i}+\nabla v_i\Big]\\ &= -\nabla P+\eta\nabla^2\mathbf{v}.\end{aligned} \tag{B2}$$

(Eq. (B1) has been used in the second step of Eq. (B2)). In Eq. (B2), $P$ is the pressure, $\eta$ is the coefficient of viscosity, and $D\mathbf{v}/Dt$ denotes the rate of change of velocity while moving with the fluid. For an area of fluid whose normal points in the i-direction, the normal force/area is -P while the shear force/area in the j-direction ($j\neq i$) is

$$f_{ij}=\eta\Big[\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\Big]. \tag{B3}$$

The sum of the pressure and viscous forces on two opposing faces of an infinitesimal cube of fluid give the pressure and viscous force/volume on the cube: these are the two terms on the right side of Eq. (B2). For our application, we assume that the fluid velocity is small enough that the left side of Eq. (B2) is negligibly small compared to either of the two terms on the right side, so (B2) becomes

$$+\eta\nabla^2\mathbf{v}=\nabla P. \tag{B4}$$

We must solve Eqs. (B1), (B4), subject to the boundary conditions.

### 2. Boundary Conditions

We shall consider that the object is a sphere of radius $R$. Far from it, the boundary conditions, which obviously satisfy Eqs. (B1), (B4), are that the fluid velocity and pressure are constant, say

$$\lim_{r\to\infty}\mathbf{v}=w\mathbf{i}_z=w[\mathbf{i}_r\cos\theta-\mathbf{i}_\theta\sin\theta],\quad \lim_{r\to\infty}P=Q.$$

Here, $\mathbf{i}_z$, $\mathbf{i}_r$ and $\mathbf{i}_\theta$ are respectively the unit vectors in the $z$-direction and in the spherical coordinate $r$ and $\theta$ directions, and $w$ is the constant asymptotic fluid speed, while $Q$ is the constant asymptotic pressure. There is no loss in generality in observing that, for $r>R$, the solution may be written

$$\begin{aligned}\mathbf{v} &= \mathbf{i}_r[w\cos\theta+v_r'(r,\theta)]+\mathbf{i}_\theta[-w\sin\theta+v_\theta'(r,\theta)],\\ P &= Q+p(r,\theta).\end{aligned}$$

where $v_r'$, $v_\theta'$ and $p$ all vanish for large $r$. Nothing depends upon the azimuthal angle $\phi$ because the solution must be rotationally symmetric about the $z$-axis.

The boundary condition at the surface $r=R$ of the sphere is that $\mathbf{v}=0$. But this is only possible if $v_r'(r,\theta)=wf(r)\cos\theta$, $v_\theta'(r,\theta)=wg(r)\sin\theta$, where $f(R)=-1$, $g(R)=1$. Any additional terms must vanish both at infinity and on the sphere's surface, but that is the solution for a fluid at rest, vanishing everywhere. Thus we look for solutions of the form

$$\begin{aligned}\mathbf{v} &= \mathbf{i}_r w\cos\theta[1+f(r)]+\mathbf{i}_\theta w\sin\theta[-1+g(r)],\\ P &= Q+p(r,\theta).\end{aligned} \tag{B5}$$

### 3. Fluid Velocity and Pressure

First, consider the constraint of Eq. (B1). Putting Eq. (B5) into Eq. (B1) gives

$$\begin{aligned}\nabla\cdot\mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}r^2\{w\cos\theta[1+f(r)]\}+\\ &\quad\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\{w\sin\theta[-1+g(r)]\}=0\end{aligned} \tag{B6}$$

(expressions for vector operators in spherical coordinates can be found, e.g., on the back cover of Jackson's book[81]). This tells us that

$$\begin{aligned}\frac{1}{r^2}\frac{d}{dr}r^2f(r)+\frac{2}{r}g(r) &= 0 \quad\text{or}\\ g(r) &= -f(r)-\frac{r}{2}f'(r)\end{aligned} \tag{B7}$$

Next we apply Eq. (B4). This can be rewritten as a first order equation in terms of $\mathbf{B}\equiv\nabla\times\mathbf{v}$, by means of the identity $\nabla\times(\nabla\times\mathbf{v})=\nabla(\nabla\cdot\mathbf{v})-\nabla^2\mathbf{v}=-\nabla^2\mathbf{v}$ (using (B1)):

$$\eta\nabla\times\mathbf{B}=-\nabla P. \tag{B8}$$

Eq. (B8) looks like Maxwell's equation for a magnetic field caused by an azimuthally symmetric current $\sim\nabla P$. Thus we expect that $\mathbf{B}\sim\mathbf{i}_\phi$, and this is borne out by calculation. Using Eqs. (B5) and (B7),

$$\begin{aligned}\mathbf{B} &\equiv \nabla\times\mathbf{v}=\mathbf{i}_\phi\frac{1}{r}\Big[\frac{\partial}{\partial r}rv_\theta-\frac{\partial}{\partial\theta}v_r\Big]\\ &= -\mathbf{i}_\phi w\frac{\sin\theta}{r}\Big[\frac{r^2}{2}f''(r)+2rf'(r)\Big].\end{aligned} \tag{B9}$$

Write Eq. (B9) as $\mathbf{B}=\mathbf{i}_\phi B(r)\sin\theta$, and insert this into Eq. (B8):

$$\begin{aligned}\nabla\times\mathbf{B} &= \Big[\mathbf{i}_r\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta-\mathbf{i}_\theta\frac{1}{r}\frac{\partial}{\partial r}r\Big]B(r)\sin\theta\\ &= \mathbf{i}_r\frac{1}{r}2\cos\theta B(r)-\mathbf{i}_\theta\sin\theta\frac{1}{r}\frac{\partial}{\partial r}rB(r)\\ &= -\frac{1}{\eta}\nabla P=-\mathbf{i}_r\frac{1}{\eta}\frac{\partial}{\partial r}p-\mathbf{i}_\theta\frac{1}{\eta r}\frac{\partial}{\partial\theta}p.\end{aligned} \tag{B10}$$

Eq. (B10) can only be satisfied if $p/\eta = h(r)\cos\theta$, where $B$, $h$ satisfy the two equations

$$B(r) = -\frac{r}{2}h'(r), \quad h(r) = -B(r) - rB'(r). \tag{B11}$$

Combining Eqs. (B11) yields the second order differential equation for $B(r)$,

$$\frac{d^2}{dr^2}rB(r) - \frac{2}{r}B(r) = 0.$$

The solutions are $B(r) = Cr^{-2}$, and $B(r) \sim r$ (which we discard). Then, from (B11), we find $h(r) = B(r)$.

Returning to Eq.(B9), we now have a differential equation for $f(r)$,

$$\frac{C}{r^2} = -\frac{w}{r}\Big[\frac{r^2}{2}f''(r) + 2rf'(r)\Big]. \tag{B12}$$

The inhomogeneous solution is $C/wr$, and the two solutions of the homogeneous equation are $C'r^{-3}$ and $\sim 1$ (which we discard). Thus we obtain $f(r)$, and then get $g(r)$ from Eq. (B7):

$$f(r) = \frac{C}{wr} + \frac{C'}{r^3}, \quad g(r) = -\frac{C}{2wr} + \frac{C'}{2r^3}.$$

Finally, the boundary conditions $f(R) = -1$, $g(R) = 1$ determine the constants:

$$C = -\frac{3wR}{2}, \quad C' = \frac{R^3}{2}.$$

Putting these results into Eq. (B5) gives the velocity and the pressure:

$$\begin{aligned} \mathbf{v} &= \mathbf{i}_r w\cos\theta\Big[1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\Big] \\ &+ \mathbf{i}_\theta w\sin\theta\Big[-1 + \frac{3R}{4r} + \frac{R^3}{4r^3}\Big] \\ P &= Q - \cos\theta\frac{3w\eta R}{2r} \end{aligned} \tag{B13}$$

#### 4. Force on Sphere

First, we calculate the force due to the pressure. This force $PdA$, on an area $dA$ of the sphere's surface, points radially inward. Because of the symmetry, the integrated force only has a net component in the $-\mathbf{i}_z$ direction. The component of the force in that direction is $PdA\cos\theta$. Therefore, the net pressure force is

$$\begin{aligned} \mathbf{F}_p &= -\mathbf{i}_z\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} R^2\sin\theta d\theta d\phi\Big[Q - \cos\theta\frac{3w\eta}{2}\Big]\cos\theta \\ &= \mathbf{i}_z 2\pi w\eta R. \end{aligned} \tag{B14}$$

Second, we calculate the viscous force/area, on an element of surface area oriented along $\mathbf{i}_r$. It is in the direction $\mathbf{i}_\theta$, and is found from Eq. (B3):

$$\begin{aligned} \sum_{i,j=1}^{3}\mathbf{i}_{ri}f_{ij}\mathbf{i}_{\theta j} &= \eta\sum_{i=1}^{3}[(\mathbf{i}_\theta\cdot\nabla v_i)\mathbf{i}_{ri} + (\mathbf{i}_r\cdot\nabla v_i)\mathbf{i}_{\theta i}] \\ &= \eta\Big[\frac{1}{r}\frac{\partial}{\partial\theta}v_r - \sum_{i=1}^{3}v_i\frac{1}{r}\frac{\partial}{\partial\theta}\mathbf{i}_{ri} + \frac{\partial}{\partial r}v_\theta - \sum_{i=1}^{3}v_i\frac{\partial}{\partial r}\mathbf{i}_{\theta i}\Big], \end{aligned}$$

where the second step uses $\mathbf{i}_r\cdot\nabla = \partial/\partial r$ and $\mathbf{i}_\theta\cdot\nabla = r^{-1}\partial/\partial\theta$ and we have differentiated by parts. Using $(\partial/\partial\theta)\mathbf{i}_r = \mathbf{i}_\theta$, and $(\partial/\partial r)\mathbf{i}_\theta = 0$ and inserting the velocity components from (B13), we obtain

$$\begin{aligned} \sum_{i,j=1}^{3}\mathbf{i}_{ri}f_{ij}\mathbf{i}_{\theta j} &= \eta\Big[\frac{1}{r}\frac{\partial}{\partial\theta}v_r + \frac{\partial}{\partial r}v_\theta - \frac{1}{r}v_\theta\Big]_{r=R} \\ &= \eta\Big[\frac{\partial}{\partial r}v_\theta\Big]_{r=R} = -\eta w\sin\theta\frac{3}{2R} \end{aligned} \tag{B15}$$

Since (B15) is the viscous force/area in the $\mathbf{i}_\theta$ direction, it must be multiplied by $-\sin\theta dA$ to get the projection of the force on a surface element in the $\mathbf{i}_z$ direction. Thus, the net viscous force is

$$\begin{aligned} \mathbf{F}_v &= \mathbf{i}_z\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} R^2\sin\theta d\theta d\phi\Big[-\eta w\sin\theta\frac{3}{2R}\Big](-\sin\theta) \\ &= \mathbf{i}_z 4\pi w\eta R. \end{aligned} \tag{B16}$$

Therefore, from Eqs. (B14) and (B16), the net force is

$$\mathbf{F} = \mathbf{F}_p + \mathbf{F}_v = \mathbf{i}_z 6\pi w\eta R.$$

Of course, here $\mathbf{v} = \mathbf{i}_z w$ is the asymptotic velocity of the water with respect to the sphere at rest, while $\mathbf{v}_s = -\mathbf{v}$ is the velocity of the sphere with respect to the asymptotically resting water, so the force on the moving sphere (Stokes' law) is

$$\mathbf{F}_s = -\mathbf{v}_s 6\pi\eta R. \tag{B17}$$

Thus, in Eq. (A6), $\beta = 6\pi\eta R$.

#### 5. Force on Ellipsoid

When the shape of the object is not a sphere, still one expects from dimensional considerations that the expression for the force is of the same form as (B17). However, the radius R is replaced by a much more complicated function of the dimensions, an effective radius. For an ellipsoid $z^2/a^2 + (x^2+y^2)/b^2 = 1$ $(a \geq b)$, moving in the $z$-direction, the result is[79]

$$\begin{aligned} R_{\text{eff}}^{-1} &= \frac{3}{8}\int_0^\infty d\lambda\frac{2a^2+\lambda}{(a^2+\lambda)^{3/2}(b^2+\lambda)} \\ &= \frac{3}{8}\left[\frac{2a^2-b^2}{(a^2-b^2)^{3/2}}\ln\frac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}} - \frac{2a}{a^2-b^2}\right]. \end{aligned} \tag{B18}$$

For $b \to a$, this becomes $a^{-1}+(2/5a^3)(a^2-b^2)+...$ which, of course, reduces to the Stokes value $a^{-1}$ when $b=a$. A good fit for $.2 \le b/a \le 1$ is $R_{\text{eff}} \approx .8b + .2a$. With $b = a/2$, (B18) gives $R_{\text{eff}} \approx .60a$.

For the ellipsoid moving in the $x$ or $y$-direction, the result is

$$R_{\text{eff}}^{-1} = \frac{3}{8}\int_0^\infty d\lambda \frac{2b^2+\lambda}{(a^2+\lambda)^{1/2}(b^2+\lambda)^2}$$
$$= \frac{3}{8}\left[\frac{a^2-(3/2)b^2}{(a^2-b^2)^{3/2}}\ln\frac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}} + \frac{a}{a^2-b^2}\right]. \quad \text{(B19)}$$

For $b \to a$, this becomes $a^{-1} + (3/10a^3)(a^2-b^2)+....$ A good fit for $.2 \le b/a \le 1$ is $R_{\text{eff}} \approx .6b + .4a$. With $b = a/2$, (B19) gives $R \approx .69a$.

The area moving through the water is, in increasing order, that of the the ellipsoid moving parallel to its long axis, the ellipsoid moving perpendicular to its long axis, and the moving sphere of radius $a$. It is reasonable that the force in these three cases is also in increasing order.

### 6. Rotational Boundary Conditions

The calculation of the viscous torque, felt by a sphere rotating with constant angular velocity through a fluid asymptotically at rest, follows the same lines as for the translating sphere, and is simpler. First comes the boundary conditions. As before, we consider the problem from the reference frame of the sphere, so that the fluid's asymptotic velocity is

$$\lim_{r\to\infty} \mathbf{v} = \mathbf{i}_\phi \Omega r \sin\theta \qquad \lim_{r\to\infty} P = Q. \quad \text{(B20)}$$

$\Omega$ is the constant asymptotic fluid angular velocity about the $z$-axis, while $Q$ is the constant asymptotic pressure. (B20) satisfies Eqs. (B1), (B4). The general solution has the form

$$\mathbf{v} = \mathbf{i}_\phi[\Omega r\sin\theta + v'_\phi(r,\theta)], \quad P = Q + p(r,\theta)$$

where $v'_\phi$ and $p$ vanish for large $r$ and satisfy Eqs. (B1), (B4).

The boundary condition at the surface $r = R$ of the sphere is that $\mathbf{v} = 0$. But this is only possible if $v'_\phi(r,\theta) = \Omega f(r) r\sin\theta$, where $f(R) = -1$. Any additional terms must vanish both at infinity and on the sphere's surface, but that is the solution for a fluid at rest, vanishing everywhere. Thus we look for solutions of the form

$$\mathbf{v} = \mathbf{i}_\phi \Omega r\sin\theta[1+f(r)], \quad P = Q + p(r,\theta). \quad \text{(B21)}$$

### 7. Fluid Velocity and Pressure

The constraint of Eq. (B1) is identically satisfied, since

$$\nabla\cdot\mathbf{v} = r\sin\theta\frac{\partial}{\partial\phi}v_\phi = 0.$$

Next we apply Eq. (B4), in the form (B8). First we calculate

$$\begin{aligned}\mathbf{B} &\equiv \nabla\times\mathbf{v}\\ &= \Big[\mathbf{i}_r\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta - \mathbf{i}_\theta\frac{1}{r}\frac{\partial}{\partial r}r\Big]\Omega r\sin\theta[1+f(r)]\\ &= 2\Omega[\mathbf{i}_r\cos\theta(1+f) - \mathbf{i}_\theta\sin\theta(1+f+\frac{r}{2}f')]. \quad \text{(B22)}\end{aligned}$$

Then, according to (B8),

$$\begin{aligned}\nabla\times\mathbf{B} &= \mathbf{i}_\phi 2\Omega\frac{1}{r}\Big[\frac{\partial}{\partial r}r[-\sin\theta(1+f+\frac{r}{2}f')]\\ &\quad -\frac{\partial}{\partial\theta}[\cos\theta(1+f)]\Big]\\ &= -\mathbf{i}_\phi 4\Omega\sin\theta[f'+\frac{r}{4}f'']\\ &= -\frac{1}{\eta}\nabla P = -\frac{1}{\eta}\Big[\mathbf{i}_r\frac{\partial}{\partial r}p + \mathbf{i}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}p\Big]. \quad \text{(B23)}\end{aligned}$$

From Eq. (B23), it follows that

$$f' + \frac{r}{4}f'' = 0, \quad p = 0.$$

The two solutions are $f(r) = C/r^3$ and $\sim 1$ (which we discard). The boundary condition $f(R) = -1$ determines the constant, $C = -R^3$. Thus, we obtain from Eq. (B21), the velocity

$$\mathbf{v} = \mathbf{i}_\phi\Omega r\sin\theta[1-\frac{R^3}{r^3}]. \quad \text{(B24)}$$

### 8. Torque on Sphere

First we find the force in the $\phi$-direction, using Eq. (B3), and then we can calculate the torque. The force/area, on the surface area $dA$ oriented along $\mathbf{i}_r$, in the direction $\mathbf{i}_\phi$ is

$$\sum_{i,j=1}^{3}\mathbf{i}_{ri}f_{ij}\mathbf{i}_{\phi j} = \eta\sum_{i,=1}^{3}[(\mathbf{i}_\phi\cdot\nabla v_i)\mathbf{i}_{ri} + (\mathbf{i}_r\cdot\nabla v_i)\mathbf{i}_{\phi i}]$$
$$= \eta\Big[\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}v_r - \sum_{i,=1}^{3}v_i\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\mathbf{i}_{ri} + \frac{\partial}{\partial r}v_\phi - \sum_{i,=1}^{3}v_i\frac{\partial}{\partial r}\mathbf{i}_{\phi i}\Big].$$

Since $v_r = 0$, $(\partial/\partial\phi)\mathbf{i}_r = \sin\theta\mathbf{i}_\phi$ and $(\partial/\partial r)\mathbf{i}_\phi = 0$, we have from (B24),

$$\sum_{i,j=1}^{3}\mathbf{i}_{ri}f_{ij}\mathbf{i}_{\phi j} = \eta\Big[-\frac{1}{r}v_\phi + \frac{\partial v_\phi}{\partial r}\Big]_{r=R} = 3\eta\Omega\sin\theta. \quad \text{(B25)}$$

The torque/area on the surface of the sphere is this force/area multiplied by the moment arm $R\sin\theta$, and it points in the $\mathbf{i}_z$ direction, so the net viscous torque is

$$\mathcal{T} = \mathbf{i}_z \int_0^{2\pi} \int_0^{\pi} \int_0^{2\pi} R^2 \sin\theta d\theta d\phi [R\sin\theta][3\eta\Omega\sin\theta]$$
$$= \mathbf{i}_z 8\pi\eta\Omega R^3.$$

Of course, here $\omega = \mathbf{i}_z\Omega$ is the asymptotic angular velocity of the water with respect to the sphere at rest, while $\omega_s = -\mathbf{i}_z\Omega$ is the angular velocity of the sphere with respect to the asymptotically resting water, so the torque on the moving sphere is

$$\mathcal{T}_s = -\omega_s 8\pi\eta R^3. \quad \text{(B26)}$$

Thus, in Eq. (A8), $\beta' = 8\pi\eta R^3$.

### 9. Torque on Ellipsoid

When the shape of the object is the ellipsoid $z^2/a^2 + (x^2+y^2)/b^2 = 1$ $(a \geq b)$, the expression for the torque is of the same form as (B26), but the radius R is replaced by another expression. For rotation about the long ($a$) axis, the result is[79]

$$R_{\text{eff}}^{-3} = \frac{3}{2}\int_0^\infty d\lambda \frac{1}{(a^2+\lambda)^{1/2}(b^2+\lambda)^2}$$
$$= \frac{3}{2}\Bigg[\frac{a}{(a^2-b^2)b^2}$$
$$- \frac{1}{2(a^2-b^2)^{3/2}} \ln\frac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}}\Bigg]. \quad \text{(B27)}$$

For $b \to a$, this becomes $a^{-3} + (6/5a^5)(a^2-b^2) + ....$ A good fit for $.2 \leq b/a \leq 1$ is $R_{\text{eff}} \approx .84b + .16a$. With $b = a/2$, (B27) gives $R_{\text{eff}} \approx .59a$.

For rotation about either of the other axes,

$$R_{\text{eff}}^{-3} = \frac{3}{2}\int_0^\infty d\lambda \frac{1}{(a^2+\lambda)^{3/2}(b^2+\lambda)^2}[\lambda + \frac{2a^2b^2}{a^2+b^2}]$$
$$= \frac{3}{2(a^2+b^2)}\Bigg[\frac{-a}{(a^2-b^2)}$$
$$+ \frac{(a^2-(1/2)b^2)}{(a^2-b^2)^{3/2}} \ln\frac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}}\Bigg]. \quad \text{(B28)}$$

For $b \to a$, this becomes $a^{-3} + (9/10a^5)(a^2-b^2) + ....$ A good fit for $.2 \leq b/a \leq 1$ is $R_{\text{eff}} \approx .56b + .44a$. With $b = a/2$, (B28) gives $R_{\text{eff}} \approx .72a$.

## APPENDIX C: GEOMETRICAL OPTICS FROM THE WKB APPROXIMATION

### 1. The Problem To Be Solved

The problem addressed in the next four appendices is to find the image of a point source of light, made by a ball lens of limited aperture. This is used to discuss the optimal choice of aperture radius.

Consider a ball lens of radius $R$ (diameter $D$) and index of refraction $n = 3/2$. ($n = 1.5$ is close enough to the BK7 glass index $n = 1.517$ of the ball lens of our experiments.) It follows from Eq. (7) that the focal length of the lens is $f = 3R/2$.

The point source of light has wave-number $k \equiv 2\pi/\lambda$, where $\lambda = .55\mu$m (green light). It is placed at the focal distance $f$ from the center of the lens. Rays pass through the lens and then through a coaxial hole of radius $b$ in a screen (the so-called exit pupil), and proceed onwards.

The light does not converge to a point at infinity, as predicted by geometrical optics for an ideal lens. Instead, the light intensity distribution which appears on a screen at infinity (placed such as to make the image as sharp as possible) is a circular blob, Although it sounds like something used by a racetrack oddsmaker, this light intensity distribution is called the *point spread function* because it describes how the light from a point source is spread out by the lens.

But, first, the connection should be made between this problem and the one we actually want to solve. The latter is to use the lens as a magnifying glass. That is, one places the point source on the optic axis slightly closer to the lens than $f$, so that the sharpest image is on a plane at 25cm on the *same* side of the lens as the source. One then divides this magnified image intensity by the lens magnification, $m \approx 25/f$ ($f << 25$cm), to obtain the apparent image intensity to scale (i.e., as if there was in fact a spread-out object of that size being precisely imaged, instead of a point source being imprecisely imaged.)

However, if the source is instead put at the focal length, with the image at $-\infty$, the angular magnification is still $m$. But, this is the same angular magnification as when the image is at $+\infty$, on the *opposite* side of the lens from the source. It is this simpler problem we are addressing.

The point spread function for this simpler problem can be readily utilized to find the intensity distribution for the magnifier application. For example, suppose we find a ring of light at infinity, with a dark boundary which makes angle $\beta_0$ with a point at the center of the lens. The magnifier usage has this circle of vanishing intensity appearing on the 25cm image plane with radius $25\beta_0$. Therefore, the apparent radius of the circle of light is $25\beta_0/m = \beta_0 f$.

### 2. Light Field in the WKB Approximation

We shall accept the argument[82] that there is no appreciable error in calculating the light intensity by taking the monochromatic light amplitude to be described by a complex scalar field $U(\mathbf{x})\exp -i\omega t$, instead of the actual vector electromagnetic field, with the time average intensity given by $|U(\mathbf{x})|^2$.

The wave equation for $U(\mathbf{x},t)$ with a point source of